Existence and uniqueness of solutions for first-order discrete systems

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Abstract

This paper deals with the existence and uniqueness of solutions to two-point boundary value problems for first-order discrete systems. The approach is based on the fixed point theorems of Perov and Schauder. The novelty of this paper is that this approach is combined with the technique that uses convergent to zero matrices and vector norms for treating discrete systems. Two examples are presented to illustrate the theory.

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1 Introduction

Let $f, g : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and consider the discrete boundary value problem

$$\frac{\Delta x_i}{h} = f(t_i, x_i, y_i), \qquad i = 0, 1, \dots, n-1,$$
$$\frac{\Delta y_i}{h} = g(t_i, x_i, y_i), \qquad i = 0, 1, \dots, n-1, \tag{1.1}$$
$$u x_0 + v x_n = w, \qquad u + v \neq 0,$$
$$\overline{u} y_0 + \overline{v} y_n = \overline{w}, \qquad \overline{u} + \overline{v} \neq 0, \tag{1.2}$$

where $h = \frac{N}{n} < N$; the grid points are denoted by $t_i = ih$ for $\Delta x_i = x_{i+1} - x_i$ for $i = 0, 1, \dots, n.; u, v, w, \overline{u}, \overline{v}, \overline{w}$ are constants.

The modeling and simulation of some nonlinear problems have aided the fast development of boundary value problem theory. The boundary value problems for differential and difference equations were extensively discussed in the literature by various methods (see, for example [1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [21], [23] and [24]).

It is of interest to note that the approach of vector-valued metrics and convergent to zero matrices for differential equations have been intensively studied in literature. In [5], Bolojan et al. used the approach of vector-valued metrics and convergent to zero matrices to establish the existence of solution for the initial value problems of nonlinear first order differential systems with nonlinear nonlocal boundary conditions of functional type. The existence results were obtained by applying the fixed point concepts of Perov, Schauder, and Leray-Schauder. Then Berrezoug et al. in [4] used the fixed point approach in vector Banach spaces to study a system of impulsive

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differential equations. The main results were proved by using Perovs and Krasnoselskii fixed point type theorems in generalized Banach spaces.

However, very little work has been done for the existence of solutions in Generalized Banach spaces for difference equations. Recently in [8], the authors used the idea of fixed point theory in generalized Banach spaces to prove the existence and uniqueness of solutions for some classes of semilinear systems of difference equations with initial and boundary conditions.

Motivated by the above work, we present a vector version of the fixed point theorem for treating systems of discrete problem (1.1), (1.2). By applying the technique vector-valued metrics and matrices convergent to zero as in [5], we obtain results that extend previous work in the area of discrete boundary value problems [7], [18], [22] and [23]. The existence result is given by means of Schauder's fixed point theorem and the existence and uniqueness of solution is obtained via a fixed point theorem due to Perov. Two examples are presented to illustrate the theory.

2 Preliminary results

In this section, we introduce some notations, definition and basic results which are used throughout this paper.

Definition 2.1. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbb{R}^{n+1}$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) = d(u, w) + d(w, v)$ for all $u, v, w \in X$

Here, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}, \mathbf{x} = (x_0, x_1, \ldots, x_n), \mathbf{y} = (y_0, y_1, \ldots, y_n)$, by $\mathbf{x} \leq \mathbf{y}$ we mean $x_i \leq y_i$ for $i = 0, 1, \ldots, n$. We call the pair (X, d) a *generalized metric space* with

$$d(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} d_0(\mathbf{x}, \mathbf{y}) \\ . \\ . \\ . \\ d_n(\mathbf{x}, \mathbf{y}) \end{pmatrix},$$

where $d_i, i = 0, 1, \ldots, n$ is a metric on X. Notice that d is generalized metric space on X if and only if $d_i, i = 0, 1, \ldots, n$ are metrics on X. For such a space convergence and completeness are similar to those in usual metrics spaces.

Definition 2.2. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \text{ as } k \to \infty.$$

The property of being convergent to zero is equivalent to each of the following conditions from the characterisation lemma below (see [2], [3], [19], [20], [25], [26]).

Lemma 2.3. Let M be a square matrix of nonnegative numbers. The following statements are equivalent:

(i) M is a matrix convergent to zero;

(ii) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);
(iii) the eigenvalues of M are located inside the unit disc of the complex plane;
(iv) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Note that, according to the equivalence of the statements (i) and (iv), a matrix M is convergent to zero if and only if the matrix $I - M$ is *inverse-positive.*

Definition 2.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq Md(x, y), \forall x, y \in X. \tag{2.1}$$

Theorem 2.5 (Schauder)**.** Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e. T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.

3 Existence results

In this section, first we show that the existence of solutions to the problems (1.1), (1.2) follows from Perov's fixed point theorem in case that the nonlinearity f, g and the functionals $a_i, b_i, i = 1, 2$ satisfy Lipschitz conditions.

Let $X := \mathbb{R}^{n+1}$. We consider the vector-valued norm

$$\| (\mathbf{x}, \mathbf{y}) \| = \begin{bmatrix} | \mathbf{x} | \\ | \mathbf{y} | \end{bmatrix}, \tag{3.1}$$

for $(\mathbf{x}, \mathbf{y}) \in X \times X$. Also $| \mathbf{x} | = \max_{i=0,\dots,n} | x_i |$ for $\mathbf{x} \in X$, and define

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| \qquad \text{for all } \mathbf{x}, \mathbf{y} \in X.$$

The pair (X, d) is called a generalized Banach space.

We can rewrite the problem (1.1), (1.2) as a system of summation equation of the form

$$\begin{aligned} x_i &= h \sum_{j=0}^{n-1} G_1(i,j) f(t_j, x_j, y_j) + \frac{w}{u+v}, \qquad i = 0, 1, \dots, n, \\ y_i &= h \sum_{j=0}^{n-1} G_2(i,j) g(t_j, x_j, y_j) + \frac{\overline{w}}{\overline{u}+\overline{v}}, \qquad i = 0, 1, \dots, n, \end{aligned} \tag{3.2}$$

where

$$G_1(i,j) = \begin{cases} \dfrac{u}{u+v} & \text{for } 0 \leq j \leq i-1, \\ -\dfrac{v}{u+v} & \text{for } i \leq j \leq n-1, \end{cases}$$

and

$$G_2(i,j) = \begin{cases} \dfrac{\overline{u}}{\overline{u}+\overline{v}} & \text{for } 0 \leq j \leq i-1, \\ -\dfrac{\overline{v}}{\overline{u}+\overline{v}} & \text{for } i \leq j \leq n-1. \end{cases}$$

It is obvious that the system (3.2) can be viewed as a fixed point problem

$$\mathbf{T}(\mathbf{x},\mathbf{y}) = (\mathbf{x},\mathbf{y}),$$

for all $(\mathbf{x},\mathbf{y}) \in X \times X$ so that

$$\mathbf{T}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} T_1(\mathbf{x},\mathbf{y})_0, \ldots, T_1(\mathbf{x},\mathbf{y})_n \\ T_2(\mathbf{x},\mathbf{y})_0, \ldots, T_2(\mathbf{x},\mathbf{y}_n \end{pmatrix}.$$

We define the operator $\mathbf{T}$ in a componentwise based on the form of (3.2), where

$$T_1(\mathbf{x},\mathbf{y})_i := h\sum_{j=0}^{n-1} G_1(i,j)\, f(t_j,x_j,y_j) + \frac{w}{u+v}, \qquad i = 0,1,\ldots,n,$$
$$T_2(\mathbf{x},\mathbf{y})_i = h\sum_{j=0}^{n-1} G_2(i,j)\, g(t_j,x_j,y_j) + \frac{\overline{w}}{\overline{u}+\overline{v}}, \qquad i = 0,1,\ldots,n,$$

for all $(\mathbf{x},\mathbf{y}) \in X \times X$. Define

$$S_l = \max \mid G_l(i,j) \mid$$

for $l = 1, 2$.

Theorem 3.1. Let $f, g : [0,N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $u + v \neq 0$, $\overline{u} + \overline{v} \neq 0$. There are constants $a_1, a_2, b_1, b_2 > 0$ such that

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq a_1|x-\overline{x}| + b_1|y-\overline{y}|, \tag{3.3}$$
$$|g(t,x,y) - g(t,\overline{x},\overline{y})| \leq a_2|x-\overline{x}| + b_2|y-\overline{y}|, \tag{3.4}$$

for all $t \in [0,N]$, $(x,y) \in \mathbb{R}^2$. In addition assume that the matrix

$$M = \begin{bmatrix} hnS_1a_1 & hnS_1b_1 \\ hnS_2a_2 & hnS_2b_2 \end{bmatrix} \tag{3.5}$$

is convergent to zero. Then the problem (1.1), (1.2) has a unique solution.

We shall apply Perov′s fixed point theorem to the problem (1.1), (1.2).

Proof. Define the operator

$$\mathbf{T} = (T_1, T_2) : X \times X \to X \times X,$$

where T_1, T_2 are given by

$$T_1(\mathbf{x},\mathbf{y})_i := h\sum_{j=0}^{n-1} G_1(i,j)\, f(t_j,x_j,y_j) + \frac{w}{u+v}, \qquad i = 0,1,\ldots,n,$$
$$T_2(\mathbf{x},\mathbf{y})_i = h\sum_{j=0}^{n-1} G_2(i,j)\, g(t_j,x_j,y_j) + \frac{\overline{w}}{\overline{u}+\overline{v}}, \qquad i = 0,1,\ldots,n,$$

for all $(x, y) \in X \times X$.

We prove that T is contractive with respect to the convergent to zero matrix M, more exactly that

$$\|\mathbf{T}(\mathbf{x},\mathbf{y}) - \mathbf{T}(\overline{\mathbf{x}},\overline{\mathbf{y}})\| \le \begin{pmatrix} hnS_1a_1 & hnS_1b_1 \\ hnS_2a_2 & hnS_2b_2 \end{pmatrix} \begin{bmatrix} |\mathbf{x}-\overline{\mathbf{x}}| \\ |\mathbf{y}-\overline{\mathbf{y}}| \end{bmatrix}.$$

We have

$$\begin{aligned} \left|T_1(\mathbf{x},\mathbf{y})_i - T_1(\overline{\mathbf{x}},\overline{\mathbf{y}})_i\right| &\le h\sum_{j=0}^{n-1} |G_1(i,j)| \left|f(t_j,x_j,y_j) - f\left(t_j,\overline{x}_j,\overline{y}_j\right)\right| \\ &\le h\sum_{j=0}^{n-1} |G_1(i,j)| \left[a_1 \mid x_j - \overline{x}_j \mid + b_1 \mid y_j - \overline{y}_j \mid\right] \\ &\le hnS_1\left[a_1 \mid \mathbf{x}-\overline{\mathbf{x}} \mid + b_1 \mid \mathbf{y}-\overline{\mathbf{y}} \mid\right] \end{aligned} \tag{3.6}$$

for $i = 0, 1, \ldots, n$. Similarly we have

$$\begin{aligned} \left|T_2(\mathbf{x},\mathbf{y})_i - T_2(\overline{\mathbf{x}},\overline{\mathbf{y}})_i\right| &\le h\sum_{j=0}^{n-1} |G_2(i,j)| \left|g(t_j,x_j,y_j) - g\left(t_j,\overline{x}_j,\overline{y}_j\right)\right| \\ &\le h\sum_{j=0}^{n-1} |G_2(i,j)| \left[a_2 \mid x_j - \overline{x}_j \mid + b_2 \mid y_j - \overline{y}_j \mid\right] \\ &\le hnS_2\left[a_2 \mid \mathbf{x}-\overline{\mathbf{x}} \mid + b_2 \mid \mathbf{y}-\overline{\mathbf{y}} \mid\right], \end{aligned} \tag{3.7}$$

for $i = 0, 1, \ldots, n$. Both inequalities (3.6) and (3.7) can be put together and be written equivalently as

$$\begin{bmatrix} \left|T_1(\mathbf{x},\mathbf{y})_i - T_1(\overline{\mathbf{x}},\overline{\mathbf{y}})_i\right| \\ \left|T_2(\mathbf{x},\mathbf{y})_i - T_2(\overline{\mathbf{x}},\overline{\mathbf{y}})_i\right| \end{bmatrix} \le \begin{pmatrix} hnS_1a_1 & hnS_1b_1 \\ hnS_2a_2 & hnS_2b_2 \end{pmatrix} \begin{bmatrix} |\mathbf{x}-\overline{\mathbf{x}}| \\ |\mathbf{y}-\overline{\mathbf{y}}| \end{bmatrix}$$

or using the vector-valued norm

$$\|\mathbf{T}(\mathbf{x},\mathbf{y}) - \mathbf{T}(\overline{\mathbf{x}},\overline{\mathbf{y}})\| \le M \begin{bmatrix} |\mathbf{x}-\overline{\mathbf{x}}| \\ |\mathbf{y}-\overline{\mathbf{y}}| \end{bmatrix}$$

with

$$M = \begin{pmatrix} hnS_1a_1 & hnS_1b_1 \\ hnS_2a_2 & hnS_2b_2 \end{pmatrix}.$$

The result follows now from Perov′s fixed point theorem. Q.E.D.

Next, we give an application of Scahuder′s fixed point theorem. We show that the existence of solutions to the problem (1.1), (1.2) follows from Scahuder′s fixed point theorem in case f, g satisfy a relaxed growth condition.

Theorem 3.2. Let $f, g : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. There are constants $a_1, a_2, b_1, b_2 > 0$ such that

$$|f(t, x, y)| \leq a_1|x| + b_1|y| + k_1, \tag{3.8}$$

$$|g(t, x, y)| \leq a_2|x| + b_2|y| + k_2, \tag{3.9}$$

for all $t \in [0, N]$, $(x, y) \in \mathbb{R}^2$. If the matrix M is given in (3.5) is convergent to zero, then the problem (1.1), (1.2) has at least one solution.

Proof. In order to apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $X \times X$ so that $\mathbf{T}(B) \subset B$. According to (3.8) and (3.9), we obtain

$$| T_1(\mathbf{x}, \mathbf{y})_i | = \left| h \sum_{j=0}^{n-1} G_1(i, j) f(t_j, x_j, y_j) + \frac{w}{u+v} \right|$$

$$\leq hnS_1 [a_1|x_j| + b_1|y_j| + k_1] + | \frac{w}{u+v} |$$

$$\leq hnS_1 [a_1|\mathbf{x}| + b_1|\mathbf{y}|] + hnS_1k_1 + | \frac{w}{u+v} |$$

for $i = 0, 1 \ldots, n$. Similarly we have

$$| T_2(\mathbf{x}, \mathbf{y})_i | = \left| h \sum_{j=0}^{n-1} G_2(i, j) g(t_j, x_j, y_j) + \frac{\overline{w}}{\overline{u}+\overline{v}} \right|$$

$$\leq hnS_2 [a_2|x_j| + b_2|y_j| + k_2] + | \frac{\overline{w}}{\overline{u}+\overline{v}} |$$

$$\leq hnS_2 [a_2|\mathbf{x}| + b_2|\mathbf{y}|] + hnS_2k_2 + | \frac{\overline{w}}{\overline{u}+\overline{v}} | .$$

$$\begin{bmatrix} |T_1(\mathbf{x}, \mathbf{y})_i| \\ |T_2(x, y)_i| \end{bmatrix} \leq M \begin{bmatrix} | \mathbf{x} | \\ | \mathbf{y} | \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix},$$

where M is given by (3.5) and is assumed to be convergent to zero, $c_0 = hnS_1k_1 + | \frac{w}{u+v} |$ and $C_0 = hnS_2k_2 + | \frac{\overline{w}}{\overline{u}+\overline{v}} |$. Next for $|\mathbf{x}| \leq R_1$ and $|\mathbf{y}| \leq R_2$, we show $| T_1(\mathbf{x}, \mathbf{y})_i | \leq R_1$, $| T_2(\mathbf{x}, \mathbf{y})_i | \leq R_2$ for $i = 0, \ldots, n$. To this end it is sufficient that

$$M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$(1 - M)^{-1} \begin{bmatrix} c_0 \\ C_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Notice that $1-M$ is invertible and its inverse $(1-M)^{-1}$ has nonnegative element since M is convergent to zero. If $B = B_1 \times B_2$, where

$$B_1 = \{\mathbf{x} \in X : |\mathbf{x}| \le R_1\}$$

and

$$B_2 = \{\mathbf{y} \in X :| \mathbf{y}| \le R_2\}$$

then $\mathbf{T}(B) \subset B$ and Schauders fixed point theorem can be applied. Q.E.D.

4 Some examples

In what follows, we give two examples that illustrate our theory.

Example 4.1. Consider the special case of (1.1), (1.2) with:

$$f(t,x,y) = \frac{1}{2}\sin x + \frac{1}{4}y + t,$$

$$g(t,x,y) = \cos\left(\frac{1}{4}x + \frac{2}{3}y\right) + t,$$

$a_1 = \frac{1}{2}$, $b_1 = \frac{1}{4}$, $a_2 = \frac{1}{4}$, $b_2 = \frac{2}{3}$, $u = 40$, $v = 60$, $\overline{u} = 30$ and $\overline{v} = 70$, $w = 25$, $\overline{w} = \frac{130}{2}$, $N = 1$, the step size $h = \frac{0.5}{n}$ where $n = 10$. We have

$$M = \begin{bmatrix} \frac{3}{20} & \frac{3}{40} \\ \frac{3}{44} & \frac{2}{11} \end{bmatrix}. \tag{4.1}$$

Since the eigenvalues of M are $\lambda_1 = 0.24$, $\lambda_2 = 0.09$, the matrix (4.1) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The associated discrete boundary value problem satisfies all conditions of Theorem 3.1 and thus has a unique solution.

Example 4.2. Consider the special case of (1.1), (1.2) with:

$$f(t,x,y) = a\sin x + \frac{1}{2}\cos y + t,$$

$$g(t,x,y) = \frac{1}{2}\sin x + ay,$$

$a_1 = |a|$, $b_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $b_2 = |a|$, $u = 1$, $v = 2$, $\overline{u} = 4$ and $\overline{v} = 2$, $w = 25$, $\overline{w} = 30$, $N = 1$, the step size $h = \frac{0.5}{n}$ where $n = 10$. We have

$$M = \begin{bmatrix} \frac{|a|}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{|a|}{3} \end{bmatrix}. \tag{4.2}$$

Since the eigenvalues of M are $\lambda_1 = \frac{|a|}{3} - \frac{1}{6}$, $\lambda_2 = \frac{|a|}{3} + \frac{1}{6}$, the matrix (4.2) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It is also known that a matrix of this type is convergent to zero if $\frac{|a|}{3} + \frac{1}{6} < 1$ (see [20]). Therefore, if $|a| < \frac{5}{2}$, the matrix (4.2) is convergent to zero and thus from Theorem 3.1 the associated discrete boundary value problem has a unique solution.

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